

# On dynamics of fermion generations

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## Abstract

The hierarchy of fermion masses and EW symmetry breaking without elementary Higgs is studied on the basis of strong gauge field distributions governing the EW dynamics. The mechanism of symmetry breaking due to quark bilinears condensation is generalized to the case, when higher field correlators are present in the EW vacuum. Resulting wave functional yields several minima of quark bilinears, giving masses of three (or more) generations. Mixing is suggested to be due to kink solutions of the same wave functional. For a special form of this mixing ("coherent mixing") a realistic hierarchy of masses and CKM coefficients is obtained and arguments in favor of the fourth generation are given. Possible important role of topological charges for CP violating phases and small masses of the first generation is stressed.

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## 1 Introduction

Despite spectacular success of the standard model (SM), the Higgs sector and the pattern of fermion masses and mixing remains mostly an unsolved issue, for the theoretical overview see [1]. In this paper we suggest a framework which might shed some light on the origin of generations, the hierarchy of fermion masses, and the Higgs problem.

The topics mentioned above are related to several related problems:

- i) Dynamical origin of Higgs sector and spontaneous symmetry breaking in the  $SU(2) \times U(1)$  sector;

- ii) Fermion generations and hierarchy of fermion mass matrices;
- iv) Origin of CP violation.

Possible solutions of the Higgs problem, different from the popular SUSY scenario, have been suggested by technicolor model [2] and by economical idea of top condensate [3,4,5], with a modern development of topcolor-assisted technicolor model [6]. There a strong interaction at high scale  $M \sim 10^{15} - 10^{16}$  GeV allows to create Higgs sector dynamically, but leaves points ii) and iv) unsolved.

A way to understand large top mass was suggested already 20 years ago [7] and developed in detailed manner since then [8,9]. The symmetry responsible for large top mass was called "flavour democracy" and considered in family space in each of the sectors (up, down and leptons) separately. The realization of this symmetry in the framework of the "flavor gauge theory" was given in [10].

The flavor-democratic scenario illustrates why in each of the sectors the mass of the third family is much larger than that in first two families and allows to connect phenomenologically the CKM mixing angles with masses [11]. However, it does not consider dynamical origin of first two families and another important hierarchy: why scales of the masses in three sectors are so much different, and inside the family the mass of top is much larger, than that of bottom and tau-lepton.

Summarizing, the problem of lower generations was not addressed. It is remarkable that masses of the first generation have much smaller scale, which might signify that internal dynamics may differ from generation to generation. Also the dynamical mechanism producing generations remains unknown. It is a purpose of present paper to suggest a possible variant of such mechanism, based on nonperturbative dynamics of the EW gauge and fermion fields. We will show below that the fermion masses due to Spontaneous Symmetry Breaking (SSB) naturally form the family structure, when higher field correlators are taken into account.

We also show that dilute topological charges in the EW vacuum may be responsible for the dynamics of the lowest fermion family. This formalism can be used for producing Higgs phenomenon in the same way as it was done in the topcolor-type models [3, 4, 5, 6].

In this way the Higgs is coupled to (and made of ) all fermions, and the scalar condensate is formed dynamically, giving mass to all quarks. The field-theoretical framework allows to consider additional contributions from topological charges creating nonzero masses for light fermions of first gen-

eration. The fermion mixing is associated with the kink solutions of the same wave functional, which connect different stationary points corresponding to generations. For a special form of the mass matrix, called the coherent mixing form, the mass eigenvalues have a pronounced hierarchy and CKM mixing coefficients are expressed via the mass ratios yielding realistic values. The neutrino mass can be considered on the same ground, including leptons and quarks symmetrically, and then the mixing, both in quark and lepton sectors, is obtained in the same way.

The plan of the paper is as follows. In Section 2 the gauge interaction at intermediate scale is introduced and resulting multifermion Lagrangian is derived. The gap equation is solved and the mass matrix is obtained and discussed in Section 3. Contributions of topological charges are given in Section 4. The problem of the fermion mixing is studied in Section 5, while the SSB and Higgs dynamics is presented in Section 6. Section 7 is devoted to a summary and possible developments of the method. Three Appendices contain an additional material for derivation of formulas in the text: Appendix 1 yields the quark Green's function in the field of topological charges; Appendices 2 and 3 describe diagonalization of the mass matrices in the case of three and four generations.

## 2 Derivation of the multi–fermion Lagrangian

The SM Lagrangian can be split in two parts,

$$L_{SM} = L_{st} + L_{Higgs}, \quad (1)$$

where  $L_{st}$  contains all kinetic parts of fermions and gauge bosons and their interaction, whereas  $L_{Higgs}$  refers to all terms where the Higgs field appears. It is our purpose, as in Refs. [3]–[5], to derive  $L_{Higgs}$  with effective Higgs field from the fields present in  $L_{st}$ , which would generate dynamically Higgs condensate, fermion masses, and mixings.

To this end, first of all, we must organize fermions into some structures which enter the fundamental Lagrangian, namely,

$$\psi \epsilon \{\psi_{Li}^A, \psi_R^A\}, \quad A = \{n, \alpha\},$$

where  $\alpha = 1, 2, 3$  refers to families and  $n = 1, 2, 3, 4$  refers to "sectors" of

fermions, which can be composed as follows

$$\psi_{Li}^{1,\alpha} = \begin{pmatrix} e_L^c \\ \nu_{eL}^c \end{pmatrix}, \begin{pmatrix} \mu_L^c \\ \nu_{\mu L}^c \end{pmatrix}, \begin{pmatrix} \tau_L^c \\ \nu_{\tau L}^c \end{pmatrix}, \psi_R^{1,\alpha} = (\nu_{eR}^c, \nu_{\mu R}^c, \nu_{\tau R}^c) \quad (2)$$

$$\psi_{Li}^{2,\alpha} = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}, \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}, \psi_R^{2,\alpha} = (e_R, \mu_R, \tau_R) \quad (3)$$

$$\psi_{Li}^{3,\alpha} = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} t_L \\ b_L \end{pmatrix}, \psi_R^{3,\alpha} = (d_R, s_R, b_R) \quad (4)$$

$$\psi_{Li}^{4,\alpha} = \begin{pmatrix} d_L^c \\ u_L^c \end{pmatrix}, \begin{pmatrix} s_L^c \\ c_L^c \end{pmatrix}, \begin{pmatrix} b_L^c \\ t_L^c \end{pmatrix}, \psi_R^{4,\alpha} = (u_R^c, c_R^c, t_R^c) \quad (5)$$

Note that considering the gauge dynamics at high scale  $M$ , one can introduce, similarly to [11], the "urfermions" with quantum numbers which are possibly different from those of final diagonalized fermions in (2-5).

Urfermions are denoted by the hat sign,  $\hat{\psi}_a^A$  and supplied by an additional index  $a$ , implying that  $\hat{\psi}$  belongs to a representation of some gauge group  $G$  operating at the scale  $M$ . We shall assume here that this group is broken at low scale and only one, the lowest mass component  $a = 1$ , should be considered for low scale dynamics. The diagonalized form of  $\hat{\psi}_{a=1}^A$  will be associated with the physical states listed in (2-5).

The destiny of higher states,  $\hat{\psi}_a, a = 2, 3..$  will be discussed elsewhere, together with a possibility that the sets  $\{a\}$  and  $\{A\}$  have a common intersection. In what follows we consider the simplest case with standard fermions listed in sectors (2-5).

The fundamental Lagrangian at high scale reads ( the Euclidean fields and metrics are used everywhere)

$$L_{high} = g \hat{\psi}_a^A(x) \gamma_\mu C_\mu^{ab}(x) \hat{\psi}_b^A(x), \quad C_\mu^{ab} = C_\mu^s T_{ab}^s. \quad (6)$$

The generating functional can be written as

$$Z = const \int D_\mu(C) D\hat{\psi} D\hat{\bar{\psi}} \exp(i \int \hat{\bar{\psi}} \hat{\partial} \hat{\psi} d^4x + \int d^4x L_{high}), \quad (7)$$

where  $D_\mu(C)$  is the integration over gauge field  $C_\mu$  with the standard weight, which may be also considered as the averaging over vacuum fields  $C_\mu$ , denoted as  $\langle \mathcal{F}(C) \rangle_C$ . In this way one can exploit the cluster expansion for

$\langle \exp(\int L_{high} d^4x) \rangle$ , namely,<sup>1</sup>

$$\begin{aligned} \langle \exp \int L_{high} d^4x \rangle &= \exp \left\{ \int J_2(x_1, x_2)(\Psi(x_1)\Psi(x_2))dx_1dx_2 \right. \\ &\quad \left. + \int J_4(x_1, \dots x_4)\Psi(x_1)\dots\Psi(x_4)dx_1\dots dx_4 + \dots \right\} = \\ &= \exp \left\{ \sum_{n=2,4,6,\dots} J_n(x_1, \dots x_n)\Psi(x_1)\dots\Psi(x_n)dx_1\dots dx_n \right\}. \end{aligned} \quad (8)$$

Here  $\Psi(x) = \hat{\bar{\psi}}(x)\gamma_\mu\hat{\psi}(x)$ , and we have suppressed spinor and group indices. Note, that only connected correlators of field  $C_\mu$  enter in  $J_n$ .

As a next step, we do the Fierz transformation, which allows us to form white bilinears, e.g. for two operators

$$(\hat{\bar{\psi}}_a^A \gamma_\mu \hat{\psi}_b^A)(\hat{\bar{\psi}}_b^B \gamma_\mu \hat{\psi}_a^B) = \sum c_i (\hat{\bar{\psi}}_a^A O_i \hat{\psi}_a^B)(\hat{\bar{\psi}}_b^B O_i \hat{\psi}_b^A), \quad (9)$$

where  $c_i = -1, +1, \frac{1}{2}, \frac{1}{2}$  for  $i = S, P, V, A$ , and anticommutation of operators  $\hat{\psi}$  is taken into account.

In a similar way one can make pairwise Fierz transformation for any  $n$  in (8), and keeping only  $S$  and  $P$  terms, one arrives at the combinations

$$\begin{aligned} \Psi(x_1)\Psi(x_2) &\rightarrow (\hat{\bar{\psi}}\gamma_5\hat{\psi})(\hat{\bar{\psi}}\gamma_5\hat{\psi}) - (\hat{\bar{\psi}}\hat{\psi})(\hat{\bar{\psi}}\hat{\psi}) = \\ &= - \left\{ (\hat{\bar{\psi}}_R\hat{\psi}_L)(\hat{\bar{\psi}}_L\hat{\psi}_R) + (\hat{\bar{\psi}}_L\hat{\psi}_R)(\hat{\bar{\psi}}_R\hat{\psi}_L) \right\} \end{aligned} \quad (10)$$

with the notation  $\Phi_{RL}(x_1, x_2) \equiv \hat{\bar{\psi}}_R^A(x_1)\hat{\psi}_L^B(x_2)$ . One can rewrite (8) as follows<sup>2</sup>

$$\begin{aligned} \langle \exp \int L_{high} d^4x \rangle_C &= \exp \left\{ - \sum_{n=2,4,\dots} \int J_n(x_1, \dots x_n) \right. \\ &\quad \left. [X(x_1, x_2)X(x_3, x_4)\dots X(x_{n-1}, x_n)] dx_1\dots dx_n \right\}, \end{aligned} \quad (11)$$

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<sup>1</sup>Note, that in general the expansion (8) is not gauge invariant. To make quark propagator  $S(x, y)$  and quark mass operator  $M(x, y)$  gauge invariant, one should consider quark accompanied by the parallel transporter  $\Phi(x, y)$ , i.e.  $\text{tr}[S(x, y)\Phi(y, x)]$ . In case of confinement this precludes definition of one-particle dynamics. Here we consider nonconfining field  $C_\mu(x)$ , and then one-particle operator  $M(x, y)$  can be made gauge invariant in the local limit,  $x \rightarrow y$ .

<sup>2</sup>Note, that to form white bilinears in a connected correlator the number of fermion transmutations is always odd, hence, the minus sign in (11).

where  $X(x, y) = \Phi_{RL}(x, y)\Phi_{LR}(y, x) + \Phi_{LR}(x, y)\Phi_{RL}(y, x)$ .

At this point one can do a bozonization trick, which we perform introducing functional representation of  $\delta$ - function [12]. In short-hand notations one has<sup>3</sup>

$$\begin{aligned} \langle \exp \int L_{high} d^4x \rangle_C &= \int D\mu D\mu^+ D\varphi D\varphi^+ \exp \left\{ -i \int \mu(\varphi - \Phi_{RL}) dx - i \int \mu^+(\varphi^+ - \Phi_{LR}) dx \right\} \times \\ &\quad \times \exp \left\{ - \sum_{n=1,4} \int J_n (\varphi\varphi^+ + \varphi^+\varphi)^{n/2} dx_1 \dots dx_n \right\}. \end{aligned} \quad (12)$$

Now one can integrate over  $D\hat{\psi}D\hat{\bar{\psi}}$  in (7), since  $\hat{\psi}$  enters in (12) only bilinearly in  $\Phi_{RL}, \Phi_{LR}$ . As a result  $Z$  in (7) acquires the form

$$Z = const \int D\mu D\mu^+ D\varphi D\varphi^+ \exp(-i(\mu\varphi) - i(\mu^+\varphi^+) + K\{\mu, \varphi\}), \quad (13)$$

where the notations are used

$$S^{-1} = i\hat{\partial} + i\frac{\mu + \mu^+}{2} + i\frac{\mu - \mu^+}{2}\gamma_5 \quad (14)$$

and

$$\begin{aligned} (\mu\varphi) &\equiv \int \mu(x_1, x_2)\varphi(x_1, x_2) dx_1 dx_2 = \\ &= V_4 \int \mu(x_1 - x_2)\varphi(x_1 - x_2) d(x_1 - x_2) = V_4 \int \mu(p)\varphi(p) \frac{d^4 p}{(2\pi)^4} \end{aligned} \quad (15)$$

$$K\{\mu, \varphi\} \equiv - \sum_n \int J_n (\varphi\varphi^+ + \varphi^+\varphi)^{n/2} dx_1 \dots dx_n + tr \ln S^{-1}. \quad (16)$$

As a next step we find the stationary points in integration over  $D\mu D\varphi D\mu^+ D\varphi^+$

$$\begin{aligned} -i\mu(x) &= \frac{-\delta K}{V_4 \delta \varphi(x)} = \sum_n n \int J_n \varphi^+(x) (\varphi\varphi^+ + \varphi^+\varphi)^{n/2-1} \delta(x_1 - x_2 - x) \frac{dx_1 \dots dx_n}{V_4} \\ -i\varphi(x) &= tr \left\{ \frac{i}{2} (1 + \gamma_5) S \right\}, \quad -i\varphi^+(x) = tr \left\{ \frac{i}{2} (1 - \gamma_5) S \right\} \end{aligned} \quad (17)$$

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<sup>3</sup>We have suppressed the  $SU(2)$  isospin subscript  $i$  in  $\Phi_{RLi}$  and below in  $\varphi_i, \varphi_i^+, \mu_i, \mu_i^+$ ; in what follows we choose the gauge with  $\varphi_1 = 0, \varphi_2 \neq 0$ .

and for  $\mu^+$  the same expression, as for  $\mu$ , follows with the replacement  $\varphi \leftrightarrow \varphi^+$ . For the solutions of (17) with  $\mu = \mu^+$ ,  $\varphi = \varphi^+$  one obtains for  $\mu(p), \varphi(p)$ , keeping only terms with  $n = 2, 4, 6$ ,

$$\begin{aligned} \mu(p) &= \xi_2 \int \frac{d^4 q}{(2\pi)^4} J_2(q) d(p-q) - \\ &- \xi_4 \int \frac{d^4 q d^4 q' d^4 s}{(2\pi)^{12}} J_4(q, q', 0) d(p-q) d(s) d(s+q') \\ &+ \xi_6 \int \frac{d^4 q d^4 q' d^4 q'' d^4 s d^4 s'}{(2\pi)^{20}} J_6(q, q', q'', 0, 0) d(p-q) d(s) d(s+q) d(s') d(s'+q''). \end{aligned} \quad (18)$$

Here  $d(k) = \frac{\mu(k)}{k^2 + \mu^2(k)}$ ,  $\xi_2 = \xi_4 = 1$ ,  $\xi_6 = \frac{3}{4}$ .

Eq.(18) is the main result of this section. Solutions  $\mu_i(p)$  define the masses of different generations,  $i = 1, 2, 3, \dots$  and will be the subject of study in the following sections. The composite scalar field  $\varphi$  and its nonlocal mass  $\mu$  play the role of the corresponding Higgs parameters of the standard model.

### 3 Qualitative analysis of the resulting equation (18)

At this point we specify the scales of nonperturbative correlators of gauge field  $C_\mu$  and denote the correlation length of the correlators  $J_n$  as  $M_n$ , so that the average value of field  $\bar{C}_n \sim \sqrt{\langle \bar{C}_\mu^2 \rangle} \sim \sqrt{\langle F_{\mu\nu}^2 \rangle M_n^2}$ ; also for simplicity we assume that the correlation length does not depend on  $n$ ,  $M_n = M$ .

Then one can introduce dimensionless quantities marked with tilde,  $\tilde{\mu}(p) = \mu(p)/M$ ,  $\tilde{p} \equiv p/M$ ,  $\tilde{q} \equiv q/M$  etc., and dimensionless kernels  $\tilde{J}_n$

$$J_2 = \bar{C}^2 M^{-4} \tilde{J}_2, \quad J_4 = \bar{C}^4 M^{-12} \tilde{J}_4, \quad J_6 = \bar{C}^6 M^{-20} \tilde{J}_6. \quad (19)$$

As a result Eq. (18) keeps its form, where all quantities are now dimensionless (with the tilde sign), and

$$\xi_2 \rightarrow \tilde{\xi}_2 = \xi_2 \bar{C}^2 / M^2, \quad \xi_4 \rightarrow \tilde{\xi}_4 = \xi_4 \bar{C}^4 / M^4, \quad \tilde{\xi}_6 = \xi_6 \bar{C}^6 / M^6. \quad (20)$$

Note, that the integration over momenta  $\tilde{p}, \tilde{q}$  is now over regions of the order of unity, while the mass eigenvalues  $\tilde{\mu} = \mu/M$  are expected to be much

less than unity,  $\tilde{\mu}(0) \ll 1$ , and  $\tilde{\mu}(\tilde{p})$  decreases with  $\tilde{p}$ . Therefore  $d(\tilde{k}) \cong \tilde{\mu}(k)/\tilde{k}^2$ , and one can extract  $\tilde{\mu}(p), \tilde{\mu}(q), \dots$  from the integrals in (18) at some average point,  $\tilde{\mu}(p_*) \rightarrow \mu_*$ . As a result one can approximate the integral equation (18) by the algebraic one:

$$\mu_* = \mu_* a_2 - \mu_*^3 a_4 + \mu_*^5 a_6 - \mu_*^7 a_8 + \dots \quad (21)$$

Here  $a_n$  are functions of  $\mu_*$ ,  $a_n(\mu^*) = a_n(0) + \mu^* a'_n(0) + \dots$  and for  $\mu_* \ll 1$  one can keep only  $a_n(0)$ , which are the numbers proportional to  $(\bar{C}/M)^n$ .

Three solutions of (21) are readily obtained, when  $a_n = 0$ ,  $n \geq 8$

$$\mu_*^2(1) = 0, \quad \mu_*^2(2), \quad \mu_*^2(3) = \frac{a_4 \pm \sqrt{a_4^2 - 4a_6(a_2 - 1)}}{2a_6}. \quad (22)$$

To obtain  $\mu_2 \ll \mu_3$  we assume that  $a_4^2 \gg 4a_6(a_2 - 1)$ , obtaining in this way

$$\mu_*^2(2) \cong \frac{a_2 - 1}{a_4}, \quad \mu_*^2(3) \cong \frac{a_4}{a_6}. \quad (23)$$

We further assume, that  $\frac{a_2 - 1}{a_4} = \nu^2 \ll 1$  and  $\frac{a_4}{a_6} \cong \nu$ . Then masses of second and third generations are

$$\mu(2) = \mu_*(2)M \cong \nu M, \quad \mu(3) \cong \sqrt{\nu}M, \quad \nu \approx \left(\frac{\mu(2)}{\mu(3)}\right)^2. \quad (24)$$

From experimental quark masses one has that  $\nu \approx 10^{-4}$  for  $(u, c, t)$  and  $\nu \approx 10^{-3}$  for  $(d, s, b)$ ; parametrically  $\nu \sim \left(\frac{M}{C}\right)^2$ , and  $M$  from (24) turns out to be  $M \sim \frac{m_3^2}{m_2} \sim 20$  TeV from the  $(u, c, t)$  sector.

To make connection with previous results in topcondensate-type models [3,4,5,6], one should neglect all  $a_n$  except  $a_2(\mu^*)$ , and returning to unscaled variables and identifying flavor-depending mass  $M^{AB} = \mu^{AB}$ , writes

$$\mu^{AB} = \frac{\bar{g}^2}{M^2} \int \frac{d^4 p}{(2\pi)^4} \mu^{AC} \left( \frac{1}{p^2 + \mu^2} \right)_{CB} \quad (25)$$

Here  $\bar{g}^2 \equiv \xi \left( \frac{\bar{C}}{M} \right)^2$  estimates the kernel  $J_2$ , and cut-off at  $p \sim M$  is assumed.

One can see, that Eq.(25) is easily diagonalized in the flavor-democratic manner and one is facing the familiar fine-tuning problem [4, 5], where  $\bar{g}^2 =$

$\bar{g}_{cr}^2 + O(m_0^2/M^2)$ . The ways to circumvent this problem are suggested in TC, ETC and TC2 models (see [6, 13] for a comprehensive review). However, in this Gaussian approximation (when only  $a_2$  is kept nonzero) it is not clear how to get three generations with distinct masses, different from flavor democratic scenario [8, 9, 10].

As was shown above this can be done keeping nonzero three coefficients:  $a_2, a_4, a_6$  and this allows to obtain three generations with masses, which can be made much different by an appropriate choice of the coefficients  $a_2, a_4, a_6$ . However, here the first generation acquires zero masses and to obtain realistic values of the masses new mechanism will be introduced in next Sections. A negative feature of the result (23) for  $\mu(3)$  is that it depends only on the field correlators via  $a_4, a_6$  etc. and at this point two questions arise: why  $a_4, a_6$  etc. should be so much different, since the expansion parameter  $(\frac{C}{M})^2$  cannot be too large for realistic field configurations; also it is not clear why  $\mu(3)$  is so much different for  $b$  and  $t$  quarks. In Section 5 we shall introduce a mechanism, which can in principle explain this high sensitivity of the quark masses and appearance of the realistic hierarchy of masses even if the coefficients  $a_n$  are of the same order of magnitude.

## 4 Topological charges in the EW vacuum

In this section we demonstrate, that an admixture of topcharges in the EW vacuum can drastically change the masses of the lowest generation.

At this point we would like to see the effects of topological charges (topcharges). We do not specify here the character of topcharges and its group assignment, assuming that in nonperturbative vacuum of field  $C_\mu$  there are group-topological conditions for existence of corresponding solutions, similar to  $SU(2)$  instanton solutions. Therefore the fields of topcharges  $A_\mu^{(i)}(x - R_i)$  in singular gauge, located at points  $R_i$ , have to be added to the fields

$$C_\mu(x) \rightarrow C_\mu(x) + \sum_{i=1}^N A_\mu^{(i)}(x - R_i). \quad (26)$$

In this case the generating functional includes averaging over topcharge positions, sizes and orientations, denoted as  $D\Omega$ ,

$$Z = \text{const} \int D\kappa(C) D\Omega D\hat{\psi} D\hat{\bar{\psi}} \exp\{\mathcal{L}\} \quad (27)$$

where  $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$ , and

$$\left. \begin{aligned} \mathcal{L}_1 &= i \int \hat{\bar{\psi}}_f \hat{\partial} \hat{\psi}_f d^4x; & \mathcal{L}_2 &= \int \hat{\bar{\psi}}_f C_\mu \gamma_\mu \hat{\psi}_f d^4x, \\ \mathcal{L}_3 &= \sum_{i=1}^N \int \hat{\bar{\psi}}_f \hat{A}^{(i)}(x - R_i) \hat{\psi}_f d^4x \end{aligned} \right\} \quad (28)$$

Now we average over the fields  $C_\mu$  as before, keeping topcharges intact, which brings in a new term, estimated at the stationary points, taking  $\mu = \mu^+$ , one has

$$\mathcal{L}_2 \rightarrow \int \hat{\bar{\psi}}_f(x) i\mu(x, y) \psi_f(y) d^4x d^4y. \quad (29)$$

As a next step we average over topcharges in the same way, as it was done in case of instantons in [14, 15, 16], see also Appendix 1. This yields the following quark Green's function (see Appendix 1 for derivation).

$$S(x, y) = S_0(x, y) + \sum_{s=1}^{\infty} u_s(x) \frac{1}{\Lambda_s - i\mu} u_s^+(y), \quad (30)$$

where  $u_s, \Lambda_s$  are the eigenfunction and the eigenvalue of the ensemble of topcharges, and  $S_0 = (-i\hat{\partial} - i\mu)^{-1}$  and  $\mathcal{L} = -\int \hat{\bar{\psi}} S^{-1} \hat{\psi} d^4x$ .

Note, that in general the sum in (30) contains  $N_0$  zero modes, corresponding to the net topcharge density  $\frac{N_0}{V_4}$ , and a region of quasizero modes. Neglecting correlations between topcharges, the eigenvalues follow the Wigner semicircle law in case of instantons [15, 16]. Going to the momentum space and averaging over topcharge positions, one has

$$S(p) = \frac{1}{\hat{p} - i\mu} + \frac{c_0(p)}{-i\mu} + \frac{c_1(p)}{\bar{\mu} - i\mu}, \quad (31)$$

where we have separated contributions of zero and quasizero modes and defined

$$c_0(p) = \frac{N_0 |\overline{u_0(p)}|^2}{V_4}, \quad c_1(p) = \frac{N |\overline{u_1(p)}|^2}{V_4}, \quad \bar{\mu} = O\left(\left(\frac{\rho}{R}\right)^4\right). \quad (32)$$

Here  $u_0(p), u_1(p)$  are zero and quasizero eigenfunctions;  $\rho$  is the average topcharge size, while  $R = \left(\frac{V_4}{N}\right)^{1/4}$  is the mean distance between topcharges.

To understand the change in the basic Eq.(18) due to topcharges and introducing  $d(p) \equiv \frac{\mu(p)}{D(p)} = \frac{\mu(p)}{p^2 + \mu^2(p)}$ , one should compute

$$\frac{\partial \text{tr}}{\partial \mu} \ln S^{-1} = -\text{tr} \left( \frac{1}{S} \frac{\partial S}{\partial \mu} \right).$$

Firstly, we simplify (31) and take  $c_1(p) \equiv 0$ . In this case one has

$$d_{top}(p) = \frac{\partial}{\partial \mu} \text{tr} \ln S^{-1} = \frac{\mu}{p^2 + \mu^2} + \frac{(c_0 p)^2}{(c_0 p)^2 + \mu^2}. \quad (33)$$

Since in (33)  $\langle p \rangle \sim M$ , and  $\mu^* = \mu/M$ , for the  $d$ -factor in (18) in case of no topological charges one has  $d_* = dM \approx \mu^*$ , which produces Eq.(21). In case, when topological charges are present, i.e.  $(c_0 \neq 0)$ , one has

$$d_*(\text{topcharge}) \equiv \bar{d} \cong \frac{\mu_*}{1 + \mu_*^2} + \frac{c_0^2}{(c_0^2 + \mu_*^2)\mu_*} \approx \mu_* + \frac{c_0^2}{(c_0^2 + \mu_*^2)\mu_*} \quad (34)$$

and Eq.(21) can be rewritten as

$$\mu_* = a_2 \bar{d} - a_4 \bar{d}^3 + a_6 \bar{d}^5 - \dots \quad (35)$$

It is easy to see, that for  $\mu_* \gg \sqrt{c_0}$  one recovers old result:  $\bar{d} \rightarrow d_* \approx \mu_*$ , with two roots given in (23). However, the zero solution for  $\mu_*$ , valid for Eq. (21), is not possible in Eq. (35). Instead, for  $\mu_* \lesssim \sqrt{c_0}$ , neglecting the terms  $a_n, n \geq 4$  in (35), one obtains one root,

$$\mu_*^2(1) \approx c_0 \left( \frac{a_2}{1 - a_2} \right)^{1/2}, \quad m_1 \approx \mu_* M \approx \sqrt{c_0} M \left( \frac{a_2}{1 - a_2} \right)^{1/4}. \quad (36)$$

Hence, the mass of the lowest generation is defined by the vacuum admixture of unbalanced zero models, i.e. by the topological charge of the vacuum. It is clear that higher order terms with  $a_4, a_6, \dots$  contribute corrections to  $\mu_*(1)$ , which are of the order of  $c_0^{3/2}, c_0^{5/2}$  etc. and can be neglected. Thus in case of topcharges one obtains a new (and nonzero) solution  $\mu_*(1)$ , while the masses of higher generations are kept intact.

In the system (35) there might be other solutions, in addition to three solutions (36) and (23), which are of the order of  $M \sqrt{\frac{a_4}{a_2}}, M \sqrt{\frac{a_6}{a_4}}$  etc. However, with discussed above assumptions,  $\frac{a_2 - 1}{a_4} \ll 1, \frac{a_4}{a_6} \ll 1$ , these roots appear to be higher than the scale  $M$  and therefore unphysical. Other and smaller roots could appear, if  $\frac{a_4}{a_2} \ll c_0, \frac{a_6}{a_4} \ll c_0$  etc., which do not seem reasonable for realistic  $c_0 \lesssim 10^{-8}$ .

One more general remark is in order. As shown in Appendix 1, the approximation used in (30), when  $\mu$  enters in the denominator of second term, is valid when the correlation length of the vacuum  $M^{-1}$  is much smaller than

the size of topcharge  $\rho$ ,  $M^{-1} \ll \rho$ . Then one can consider (nonlocal) effective mass  $\mu$  as a constant inside the topcharge field. It is clear, that in general the high-n correlator term  $J_n(x_1, \dots x_n)$  will be outside of topcharge size  $\rho$  for high enough  $n$ , hence, zero mode contribution (second term in (34)) will be effectively damped in high-n terms  $a_n \bar{d}^n \rightarrow a_n \mu_*^n$ . Therefore one cannot expect additional roots from higher  $a_n \bar{d}^n$  terms, which are not present in the  $a_n \mu_*^n$  series (21). Hence we except, that topcharges can create small masses of the lowest generation, with the scale proportional to the concentration of topcharges.

## 5 Mixing due to fermion bilinear condensation. The mechanism of coherent mixing

Till now we have disregarded the matrix nature of  $\mu_{ik}, \varphi_{ik}$ . As it is clear from (9)-(12),  $\Phi_{RL}(x_1, x_2) \equiv \hat{\psi}_R^A(x_1) \hat{\psi}_L^B(x_2)$  is a matrix in the indices  $A = (n, \alpha)$ ,  $B = (n', \beta)$ , where  $n, n'$  refer to families and  $\alpha, \beta$  to fermion sectors. As will be shown below, the sector mixing in  $\alpha, \beta$  does not occur from the effective Lagrangian  $K\{\mu, \varphi\}$ , Eq. (16), but the family mixing does occur. For notational convenience in the matrices  $\hat{\mu}_{nn'}, \varphi_{nn'}$ , instead of  $n, n'$  we shall denote the family numbers  $i, k$  as  $\mu_{ik}$ ,  $i\hat{\varphi}_{ik}$ ,  $i, k = 1, 2, 3, \dots$ , and use the relation

$$\mu_{ik}(p) = (p^2 + \hat{\mu}^2)_{il} \varphi_{lk}(p), \quad d_{ik}(p) = \hat{\varphi}_{ik}(p). \quad (37)$$

Then the basic equation (18) can be rewritten in the  $x$ -space as (for  $|\hat{\mu}^2(p)| \ll p^2$ ),

$$-\square \hat{\varphi}(x) = -\frac{\delta U\{\hat{\varphi}\}}{\delta \hat{\varphi}(x)} = \tilde{J}_2 \hat{\varphi} - \tilde{J}_4 \hat{\varphi}^3 + \tilde{J}_6 \hat{\varphi}^5 - \dots, \quad (38)$$

where  $\tilde{J}_n$  are actually nonlocal kernels in  $x$ -space corresponding to the  $p$ -space kernels in (18). Also the functional  $U\{\hat{\varphi}\}$  can be written as

$$U\{\hat{\varphi}\} = -\frac{1}{2} \tilde{J}_2 \hat{\varphi}^2 + \frac{1}{4} \tilde{J}_4 \hat{\varphi}^4 - \frac{1}{6} \tilde{J}_6 \hat{\varphi}^6 + \dots \quad (39)$$

Our functional  $U\{\hat{\varphi}\}$ , Eq. (39) has the standard form which was already investigated in the local limit in search for solitonic solutions [17, 18]. As was discussed above,  $U\{\hat{\varphi}\}$  has several stationary points, which we associate with the  $p$ -space solutions of Eq. (18) or the  $x$ -space solutions, of Eq. (38). These

solutions are not like solitons, but rather solutions of nonlinear nonlocal integral (or integro-differential) equation, where the values of  $\varphi_{ik}$  are varying in the region around the stationary point  $\hat{\varphi}_n(x)$  of  $U\{\hat{\varphi}\}$ . Therefore one can identify the diagonal elements of the matrix function  $\hat{\varphi}(x)$  with  $\varphi_n(x)$  as follows

$$\hat{\varphi}_{ik}(x) = \delta_{in}\delta_{kn}\varphi_n(x), \quad n = 1, 2, 3, \dots \quad (40)$$

Let us now turn to nondiagonal elements of  $\hat{\varphi}_{ik}(x)$ . From physical point of view, since the diagonal elements are associated with stationary points of  $U\{\hat{\varphi}\}$ , nondiagonal elements  $\hat{\varphi}_{ij}$  should be solutions connecting two stationary points  $i, j$ , i.e. solutions of kink type. A well-known example of the kink for the functional  $U(\hat{\varphi}) = \frac{\lambda}{4}(\hat{\varphi}^2 - m^2/\lambda)^2$  is given by the solution  $\hat{\varphi}_{kink}(x) = \frac{m}{\sqrt{\lambda}}\tanh\left(\frac{m}{\sqrt{2}}(x - x_0)\right)$ , which connects stationary points  $\hat{\varphi}_1 = \frac{m}{\sqrt{\lambda}}$  and  $\hat{\varphi}_2 = -\frac{m}{\sqrt{\lambda}}$ . Therefore we shall assume here, that similar solutions exist in our case for Eq. (38) with the value of  $\hat{\varphi}_{ik}(x)(i \neq k)$  varying in the region between stationary "points"  $\varphi_n(x)$  with  $n = i$  and  $n = k$ .

Now coming back to qualitative discussion (in Section 3) of possible solutions  $\mu(p)$  of Eq. (18) or  $\hat{\varphi}_{ik}(x)$  of Eq.(38), one expects the average fermion masses  $\mu_{ik}(p) \rightarrow \mu_{ik}$  to be equal to  $\mu(n)$  for  $i = k = n$ , and some average of  $\mu(i)$  and  $\mu(k)$  for  $i \neq k$ . In what follows we take  $\mu_{ik}$  for the kink solution as the "geometrical average"

$$\mu_{ik} = \sqrt{\mu(i)\mu(k)} \quad (41)$$

and in this case the spectrum with all eigenvalues, with an exception of the largest one, appear to be arbitrarily small. We call this phenomenon the Coherent Mixing Mechanism (CMM).

To study qualitatively the CMM in more simple case of two families, we start with the mass matrix  $\mu_{ik}$  both in up and down sectors:

$$\mu_{ik} = \begin{pmatrix} \mu_1 & \mu_{12} \\ \mu_{12} & \mu_2 \end{pmatrix}. \quad (42)$$

The eigenvalues of  $\mu_{ik}$  in the case with  $\mu_2 \gg \mu_1$  are

$$m_+ = \mu_2 + \frac{\mu_{12}^2}{\mu_2}, \quad m_- = \mu_1 - \frac{\mu_{12}^2}{\mu_2}. \quad (43)$$

For the choice (41) and  $\mu_{12} = \sqrt{\mu_1\mu_2}$ , one has  $m_- = 0$ ,  $m_+ = \mu_1 + \mu_2$  and the CKM matrix  $V_{ik}$  has the form

$$\hat{V} = W_u W_d^+ = \begin{pmatrix} 1 - \frac{\bar{\eta}^2}{2} & \bar{\eta} \\ -\bar{\eta} & 1 - \frac{\bar{\eta}^2}{2} \end{pmatrix} \quad (44)$$

with  $\bar{\eta} \cong \sqrt{\frac{\mu_1}{\mu_1 + \mu_2}}$ , which refer to the sector  $(d, s, b)$ , since  $(u, c, t)$  sector yields much smaller  $\eta$ . It is clear, that varying  $\mu_{12}$  around the value  $\sqrt{\mu_1 \mu_2}$ , one obtains physical values of  $m_1$  in the region from  $\mu_1$  to zero, and hence  $\mu_1$  can be  $\mu_1 \gtrsim m_d$ , while  $\mu_2 \approx m_+ \sim m_s$ , and  $\eta$  has a reasonable value,  $\bar{\eta} \approx 0.2$ , in agreement with experimental data. Note, that in CMM the choice (41) gives the maximal value of mixing for fixed  $\mu_1, \mu_2$  and minimal value of  $m_- = 0$ .

For three or more generations one can write the general CMM matrix  $\hat{\mu}$  as follows<sup>4</sup>

$$\hat{\mu} = \begin{pmatrix} \mu_1 & \mu_{12} & \mu_{13} & \dots \\ \mu_{12} & \mu_2 & \mu_{23} & \dots \\ \mu_{13} & \mu_{23} & \mu_3 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \quad \mu_{ik}^2 = \mu_i \mu_k (1 + \eta_{ik}). \quad (45)$$

Here we denote a diagonal effective mass  $\mu_{ii}$  as  $\mu_i$ ; it corresponds to the  $i$ -th minimum of the functional  $U(\hat{\varphi})$ , and then consider  $\eta_{ik}$  small,  $|\eta_{ik}| \ll 1$ <sup>5</sup>.

We assume that  $\mu_1 < \mu_2 < \mu_3 < \dots$  and  $\mu_i$  not necessarily much smaller than  $\mu_{i+1}$ .

The eigenvalue equation in case of tree generations,  $\det(\hat{\mu} - \hat{m}I) = 0$ , given in Appendix 2, Eq.(A2.3), can be written as

$$m^3 - m^2\sigma + m\xi - \zeta = 0 \quad (46)$$

where coefficients can be expanded in powers of  $\eta_{ik}$ , yielding  $\sigma = \sum_i \mu_i$ ,  $\xi = -\sum_{i \neq j} \mu_i \mu_j \eta_{ij}$ ,  $\zeta = \mu_1 \mu_2 \mu_3 \left( -\frac{1}{4} \sum_{i \neq j} \eta_{ij}^2 + \frac{1}{2} \sum_{i \neq j, l \neq k} \eta_{ij} \eta_{lk} \right)$ . Since  $m_1 m_2 m_3 = \zeta$ , for very small  $\zeta$  the lowest eigenvalue  $m_1$  tends to zero, and one obtains the following hierarchy of eigenvalues

$$m_1 \cong \frac{\zeta}{\xi}, \quad m_2 \approx \frac{\xi}{\sigma}, \quad m_3 \approx \sigma. \quad (47)$$

Note, that one can easily adjust the strong hierarchy, namely,  $m_1 \ll m_2 \ll m_3$  by a simple variation of the parameters  $\eta_{ij}$ . For example, as shown in Appendix 2 for the choice  $\eta_{23} = \eta_{13} \equiv -\eta$ ,  $\eta_{12} = -\delta$ ,  $0 < \delta \ll \eta$ , one has

$$m_1 \approx \mu_1 \delta, \quad m_2 \approx \frac{\mu_3}{m_3} (\mu_1 + \mu_2) \eta, \quad m_3 \approx \mu_1 + \mu_2 + \mu_3. \quad (48)$$

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<sup>4</sup>Our input mass matrix contains  $\mu_i, \mu_{ik}$ , and the final diagonalized mass matrix contains physical masses  $m_i, i = 1, 2, 3$  ( $m_u, m_c, m_t$  and  $m_d, m_s, m_b$ ).

<sup>5</sup>Note, that for  $\mu_i = \mu_0$ , ( $i = 1, 2, 3$ ) and  $\eta_{ik} = 0$  one recovers the "ultrademocratic" mass matrix [7] -[9].

It is interesting, that for  $\delta \ll \eta \ll 1$  the CMM has made the hierarchy much more pronounced, than original situation with  $\mu_1 < \mu_2 < \mu_3$ , and  $m_1, m_2$  can be made very close to zero, while  $m_3$  is not far from  $\mu_3$ .

The mass matrix (45) is diagonalized, as shown in Appendix 2, with the help of the unitary matrix  $W$ , Eq.(A2.13), where we also introduced imaginary parts in  $\mu_{ik}$  as  $\mu_{ik} = |\mu_{ik}|e^{i\delta_{ik}}$  to account for a possible CP violation with the condition  $\delta_{12} - \delta_{13} + \delta_{23} = 0$ . A very convenient way of constructing unitary matrices  $W$  in terms of  $\mu_{ik}$  and  $m_i$  was given in [18]. There a simplified form was obtained in case  $W_{13} \rightarrow 0$ . In our specific case of CMM matrix (45), the element  $W_{31}$  tends to zero when  $\delta \rightarrow 0$ , and another simple form for  $W$ , shown in (A2.15), occurs. The resulting CKM matrix  $\hat{V}$  is readily computed from  $W_u$  and  $W_d$  Eq.(A2.15) and it is given in the Appendix 2, Eq.(A2.17). It is expressed in terms of the phases  $\delta_{ik}$  and the values  $m_i, \mu_i$  only. The latter enter via sines and cosines defined in (A2.16).

Using (A2.17) we can write a simplified version of CKM matrix realizing, that all cosine factors are equal to unity within  $(1 \div 2)\%$ , while the sine factors  $\sim 0.1$ , and one has the following estimates from PDG [19]

$$|V_{ud}| = |c_\alpha^u c_\alpha^d + s_\alpha^u s_\alpha^d e^{i\Delta_{12}}| = 0.97418 \pm 0.00027 \quad (49)$$

$$|V_{us}| \cong |s_\alpha^u e^{i\delta_{12}^u} - s_\alpha^d e^{i\delta_{12}^d}| = 0.2255 \pm 0.0019 \quad (50)$$

$$|V_{cd}| \cong |s_\alpha^d e^{-i\delta_{12}^d} - s_\alpha^u e^{-i\delta_{12}^u}| = 0.230 \pm 0.011 \quad (51)$$

$$|V_{cs}| = |c_\alpha^u c_\beta^u c_\alpha^d c_\beta^d + O(ss)| = 1.04 \pm 0.06 \quad (52)$$

$$|V_{tb}| = |c_\beta^u c_\beta^d + O(ss)| > 0.74 \quad (53)$$

$$|V_{ub}| = |s_\alpha^u (s_\beta^u e^{i\delta_{13}^u} - s_\beta^d e^{i(\delta_{12}^u + \delta_{23}^d)})| = (3.93 \pm 0.36)10^{-3} \quad (54)$$

$$|V_{cb}| = |s_\beta^u e^{i\delta_{23}^u} - s_\beta^d e^{i\delta_{23}^d}| = (4.12 \pm 0.11)10^{-2} \quad (55)$$

$$|V_{ts}| = |s_\beta^d e^{-i\delta_{23}^d} - s_\beta^u e^{-i\delta_{23}^u}| = (3.87 \pm 0.23)10^{-2} \quad (56)$$

$$|V_{td}| = |s_\alpha^d (s_\beta^d e^{-i\delta_{13}^d} - s_\beta^u e^{-i(\delta_{23}^u + \delta_{12}^d)})| = (8.1 \pm 0.6)10^{-3}. \quad (57)$$

One can see, that two equalities arise from the above expressions

$$V_{us} = -V_{cd}^*, \quad V_{ts} = -V_{cb}^* \quad (58)$$

The experimental values for moduli of these expressions are equal within the errors.

The angles  $\alpha, \beta, \gamma$  are easily computed from the entries of (49)-(57), namely:

$$\alpha = \Delta_{12}, \quad \beta = \arg(s_\alpha^d - s_\alpha^u e^{-i\Delta_{12}}), \quad \gamma = \pi - \Delta_{12} - \beta. \quad (59)$$

Assuming  $\alpha \approx \frac{\pi}{2}$  [19], our prediction is  $\beta \cong 24^\circ$ , yielding  $\sin 2\beta \cong 0.73$  not far from experiment.

Moreover, dividing (54) by (56) and using the condition  $\delta_{13} - \delta_{12} = \delta_{23}$ , one obtains  $s_\alpha^u \cong 0.1$ , which yields  $\mu_u \approx 20$  MeV, whereas dividing  $V_{td}$  (57) by (56), one obtains  $s_\alpha^d \approx 0.21$ , which defines  $\mu_d \approx 10$  MeV (both at the scale of 2 GeV). An important check is the reparametrization -invariant quantity  $\bar{\rho} + i\bar{\eta} = -\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}$ , which in our simplified ansatz (A2.15) is

$$\bar{\rho} + i\bar{\eta} = \frac{s_\alpha^u c_\alpha^d (s_\alpha^u - s_\alpha^d e^{-i\Delta_{12}})}{(s_\alpha^u)^2 + (s_\alpha^d)^2} \approx 0.18 + i0.38.$$

The assumption  $\Delta_{23} \equiv 0$  makes possible a strong cancellation in  $(s_\beta^d - s_\beta^u e^{-\Delta_{23}}) \approx (s_\beta^d - s_\beta^u)$  which should be as small as 0.04, yielding  $\mu_s \approx 0.35$  GeV,  $\mu_c \approx 14$  GeV. Using (49)-(57) one has a reasonable estimate for  $J$ , Eqs. (A2.18), (A2.19), if  $s_\beta^d$  and  $s_\beta^u$  satisfy (56),

$$J = s_\alpha^u s_\alpha^d |s_\beta^d - s_\beta^u e^{i\Delta_{23}}|^2 \sin \Delta_{12} \approx 3.2 \cdot 10^{-5} \sin \Delta_{12},$$

which agrees well with the PDG value [19]. Thus all phenomenological tests are passed by our representation.

At this point one could estimate more systematically the input values of  $\mu_{ik}$  necessary to satisfy experimental data for  $V_{ik}$ . We shall not do it here and in the next Section we present the arguments, that this procedure is better to be used in case of four generations.

In Appendix 3 we explain that in CMM the eigenvalues of four generations obey the same pattern as in the case of three generations - with a steady highest mass and volatile lower masses.

## 6 Spontaneous symmetry breaking, the Higgs condensate, and the fourth generation

The fermion bilinear condensation discussed above plays the same role in the EW spontaneous symmetry breaking, as the standard Higgs mechanism, and in this sense is an extension of the topcondensate mechanism to the case of many generations. Since there are many condensates ,  $\langle \hat{\varphi}_{ij} \rangle$  and  $\langle \hat{\varphi}_{ij}^+ \rangle$ , one might worry about multiple composite Higgs bosons and in this Section we shall study the situation with the Higgs condensate (and hence with the  $W$  and  $Z$  masses) and the scalar excitation of the condensate ("the mass of Higgs boson").

To obtain the Higgs condensate, one can use the Pagels-Stokar relation [20], where the whole spectrum of quarks in the quark loop diagrams for the scalar current is introduced. In the leading order it gives

$$v^2 = \frac{N_c}{4\pi^2} \sum_i \mu^2(i) \ln \frac{M}{\mu(i)}, \quad (60)$$

where  $v = 246$  GeV is the standard value of the Higgs condensate, and the sum is over all quark masses  $\mu(i)$  in  $n$  generations. With the hierarchy already known for three generations and assumed for four generations, the dominant contribution comes from the highest mass  $\max \mu(i) = \mu_{\max}$ , and from (60)  $\mu_{\max}$  for a given  $M$  can be estimated. The results are given in the Table.

For  $M_W$  and  $M_Z$  one has the standard relations

$$M_W^2 \cong \frac{g_2^2}{4} v^2, \quad M_Z^2 \cos^2 \Theta_W \cong \frac{g_2^2}{4} v^2. \quad (61)$$

Table. The values of highest quark masses for different mass scales  $M$  from Eq.(60)- in second row, and from [21]–in bottom row.

$M$ GeV	$10^{19}$	$10^{17}$	$10^{13}$	$10^9$	$10^5$	$10^4$	$5 \cdot 10^3$
$\mu_{\max}$ GeV	143	153	180	228	377	518	616
$\mu_{\max}$ GeV	253	259	279	316	446	591	-

Note, that (60) is a naive approximation with  $\mu(i)$  as a constant, which does not depend on momenta of loop integration. Taking that into account one obtains much higher values of  $\mu_{\max}$  for a given  $M$  [21] ( see Table), hence,  $\mu_{\max}$  cannot be associated with the top quark mass (at least in this approximation). For more discussion see the review by Cvetič [5] and Refs. therein.

For  $M$  smaller than  $5 \cdot 10^3$  GeV the Eq.(60) is not a good approximation and one should use a function  $\mu_i(p)$  as in Eq. (18), however, the consistency of the whole approach is questionable for  $\mu_{\max} \sim M$ . Therefore we consider  $\mu_{\max} \sim 0.5 \div 0.6$  TeV as the maximal value of the  $t'$  mass, and it is clear that  $t'$  mass should belong to the 4th generation of quarks.

As we noted in Section 5, the CMM ensures volatility of the masses of all lower generations, while for the highest generation the masses are rather stable. This picture is consistent with what should happen to the masses of the 3d generation, when there exists the fourth generation. Indeed,  $m_b \approx O(4$  GeV) and  $m_t \approx O(180$  GeV) are very different, and might be seen as a subject to large changes, when small changes in the mixing coefficients  $\eta_{ik}$  in Eq.(45) are made. In this way assuming existence of four generations one might resolve the old problem in the top condensate mechanism.

It is remarkable, that the analysis of precision data in [22] for  $m_{b'} = m_{t'} = 300$  GeV and the mass of heavy charged lepton  $m_E = 200$  GeV shows the same  $\chi^2$  minimum for four generations as for three, supporting in this way the idea of the fourth generation. Note also, that for degenerate  $b'$  and  $t'$  quarks the mass of each of them is roughly by ( $35 \div 40\%$ ) less than shown in the Table. For discussion of the possible parameter space of the fourth generation see [23].

Let us now discuss the topic of the possible (composite) Higgs boson mass. It is clear, that the induced Higgs Lagrangian should have the same form as in the topcondensate case, where the Higgs field  $h$  is the deflection from the stationary point  $i = j$  of the effective potential  $h = H - v \equiv \mu_{ii} - v$

$$V(H) = -m_H^2 H^+ H + \frac{\gamma_4}{2} (H^+ H)^2 + \frac{\gamma_6}{3} (H^+ H)^3 + \dots - (\bar{\psi}_{Li} \psi_{Ri} H + h.c.). \quad (62)$$

The difference from the standard topcondensate case is that i) there only one minimum of  $V(H)$  exists at  $H = v$ , while in our case at least three minima should be present; ii) higher order terms in  $h$  are present already at the tree level, e.g.  $\gamma_4(\text{tree}) \neq 0$  due to higher correlator terms  $J_n, n \geq 4$  , while for the topcondensate case (only  $J_2$  present) these terms are induced by fermion

loop diagrams. Moreover the higher correlator terms contribute dominantly to the coefficient of  $h^2/2$  (the Higgs mass) in the situation discussed in Section 3 (e.g.  $a_6 \gg a_4 > a_2$  etc.). Hence, we can conclude that in this case the lowest Higgs mass appears for the largest minimum, i.e. near  $\mu_{ii} = \mu_{\max}$ . Then (also neglecting admixture of lower minima) all coefficients are mostly quark loop-induced and hence,  $m_{Higgs} \cong 2\mu_{\max}$ , as well known [20], [4]. In our favored case of four generations, it means that the Higgs mass is around 1 TeV (for  $M \sim 10^4 - 10^5$  GeV). As noted in [22], this situation of high Higgs mass and one extra generation does not contradict precision data.

One should also have in mind, that this Higgs boson is not elementary and, if it exists at all, can be associated with an excited unstable  $q\bar{q}$  state. For more discussion of the new physics with Higgs and the fourth generation see [24].

## 7 Conclusions and outlook

Results of the paper are threefold. First of all, we present a possible dynamical scheme of  $q\bar{q}$  pair condensation which might explain the generation structure of the fermion hierarchy.

Secondly, we have found in the same scheme a possible source of fermion mixing, identifying it with the kink-type (soliton-type) solutions of the same effective potential.

Finally, we have suggested a new type of mixing pattern, called the coherent mixing mechanism (CMM), which follows from two first solutions. In this way one obtains a simple parametrization of the CKM matrix  $\hat{V}$ : in terms of two phases and the input masses  $\mu_i$  (corresponding to minima of effective potential), and resulting physical masses  $m_i$ .

In CMM strong mass shifts ( $\mu_i - m_i$ ) of all lower generations are caused by tiny changes in mixing parameters, which may explain large mass differences in  $u$  and  $d$  sectors. The same volatility in the masses for the third generation and experimental values of  $V_{ik}$  strongly prefer the scheme of four generations with  $m_t \sim m_b \sim O(300 \div 500 \text{ GeV})^6$ . In this case also the Higgs condensate and  $W, Z$  masses are correctly reproduced. The method of the paper allows to predict more explicitly the resulting masses  $m_t, m_b$  and some mixing coefficients, which is planned for future publications.

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<sup>6</sup>Four generations were found favorable also in the context of flavor democracy scenario [25].

As an additional possibility the role of topcharges in the EW vacuum is studied and shown to be effective in producing very small masses of the first generation; also the origin of the CP-violating phases can be directly connected to the topcharge contents of the EW and GUT vacuum, as was suggested before in [26] in another framework.

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## Appendix 1

### The quark Greens's function in presence of topological charges

One starts with the gauge field of the form

$$C_\nu(x) = B_\nu(x) + \sum_{i=1}^N A_\nu^{(i)}(x) \quad (\text{A1.1})$$

where  $B_\nu$  are nontopological fields, while  $A_\nu^{(i)}(x)$  are fields of topcharges, exact form of those is not important for us, but we shall assume that topcharges form a dilute gas with no correlations. Our goal is the calculation of the full quark Green's function  $S$  in the field  $C_\nu(x)$ ,

$$S = \left( -i\hat{\partial} - g\hat{B} - g \sum_{i=1}^N \hat{A}^{(i)} \right)^{-1} \quad (\text{A1.2})$$

in terms of individual topcharge Green's functions  $S_i = (-i\hat{\partial} - gB - g\hat{A}^{(i)})^{-1}$  and  $S_0 = (-i\hat{\partial} - g\hat{B})^{-1}$ .

One can use the same technic as exploited in the Faddeev-type decomposition of the Green's function for particle scattering on many centers [14]. Introducing  $t$ -matrices,  $t_i \equiv S_0 - S_i$  and amplitudes for scattering  $Q_{ik}$ , where  $i$  refers to the first and  $k$  – to the last scattering center, one has equations for  $Q_{ik}$

$$Q_{ik} = t_i \delta_{ik} - t_i S_0^{-1} \sum_{j \neq i} Q_{jk}. \quad (\text{A1.3})$$

For  $S_i$  one can use the spectral representation

$$S_i(x, y) = \sum_n \frac{u_n^{(i)}(x) u_n^{(i)+}(y)}{\mu_n^{(i)} - im}, \quad (\text{A1.4})$$

where we have introduced the quark mass  $m$  for future convenience, and  $u_n^{(i)}(x)$  satisfies equation

$$-\gamma_\mu(\partial_\mu - igB_\mu - igA_\mu^{(i)}(x))u_n^{(i)}(x) = \mu_n^{(i)}u_n^{(i)}(x). \quad (\text{A1.5})$$

Using (A1.4) one can rewrite (A1.3) as follows

$$\begin{aligned} Q_{ik}(x, y) &= \delta_{ik} \left[ S_0(x, y) - \sum_n \frac{u_n^{(i)}(x) u_n^{(i)+}(y)}{\mu_n^{(i)} - im} \right] + \\ &+ \int d^4z \sum_n \frac{u_n^{(i)}(x) u_n^{(i)+}(z)}{\mu_n^{(i)} - im} (-i\hat{\partial} - g\hat{B} - \mu_n^{(i)}) \sum_{j \neq i} Q_{jk}(z, y). \end{aligned} \quad (\text{A1.6})$$

Solving (A1.6) for  $Q_{ik}$ , one immediately finds  $S$ ,

$$S = S_0 - \sum_{i,k=1}^N Q_{ik}. \quad (\text{A1.7})$$

One can represent  $Q_{ik}$  as follows

$$Q_{ik}(x, y) = \sum_{n,n'} \frac{u_n^{(i)}(x) R_{nn'}^{ik} u_{n'}^{(k)+}(y)}{(\mu_n^{(i)} - im)(\mu_{n'}^{(k)} - im)} \quad (\text{A1.8})$$

and for  $i \neq k$ , Eq.(A1.6) reduces to (in matrix notations for upper and lower indices)

$$\hat{R} = \hat{\xi} \hat{R}, \quad (\text{A1.9})$$

where we have defined

$$\xi^{ik}_{nn'} = \frac{\int u_n^{(i)}(z)(-i\hat{\partial} - g\hat{B} - \mu_n^{(i)})u_{n'}^{(k)}(z)d^4z}{\mu_{n'}^{(k)} - im} \equiv \frac{V_{nn'}^{ik}}{\mu_{n'}^{(k)} - im}, \quad i \neq k \quad (\text{A1.10})$$

while by definition  $\xi_{nn'}^{ii} = 0$ . Introducing notation

$$\eta_{nn'}^{(i)} \equiv \mu_n^{(i)} - im)(\mu_{n'}^{(i)} - im) \int u_n^{(i)+}(x)S_0(x, y)u_{n'}^{(i)}(y)dxdy - (\mu_n^{(i)} - im)\delta_{nn'}, \quad (\text{A1.11})$$

for  $R^{ii}$  one obtains

$$R_{nn'}^{ii} = \eta_{nn'}^i + \xi_{nm}^{ij}R_{mn'}^{ji}. \quad (\text{A1.12})$$

One can rewrite (A1.9), (A1.12) as

$$\hat{R} = \hat{\eta} + \xi \hat{R}, \quad \hat{R} = \frac{1}{1 - \hat{\xi}}\hat{\eta}. \quad (\text{A1.13})$$

Denoting  $\hat{\xi} \equiv \hat{V}_{\hat{\mu}-im}$ , one can write finally  $S$  in (A1.7) as

$$S(x, y) = S_0(x, y) + \sum_{ik, nn'n''} u_n^{(i)}(x) \left( \frac{1}{\hat{\mu} - im - \hat{V}} \right)_{nn'}^{ik} \left[ \delta_{n'n''} + (\mu_{n'}^{(k)} - im)V_{n'n''}^{ii} \right] u_{n''}^{(k)}(y) \quad (\text{A1.14})$$

(Note, that  $(\hat{\mu})_{nn'}^{ik} = \mu_n^{(i)}\delta_{ik}\delta_{nn'}$ ).

Neglecting the second term in the square brackets in (A1.14) (which is reasonable for zero and quasizero eigenvalues  $\mu_n$ ), one can write

$$S(x, y) = S_0(x, y) + \sum_{ik, nn'} u_n^{(i)}(x) \left( \frac{1}{\hat{\mu} - im - \hat{V}} \right)_{nn'}^{ik} u_{n'}^{(k)+}(y) \quad (\text{A1.15})$$

One can find the eigenvalues  $\Lambda_S$  of the operator  $\hat{\mu} - im\hat{V}$ , and eigenfunctions  $u_s(x)$ , and as a result (A1.15) turn out to be

$$S(x, y) = S_0(x, y) + \sum_s u_s(x) \frac{1}{\Lambda_s - im} u_s^+(y) \quad (\text{A1.16})$$

and  $u_s(x)$  are collectivized eigenfunctions of the gas of top charges

$$\frac{1}{\hat{\mu} - im - \hat{V}} = U \frac{1}{\hat{\Lambda} - im} U^+; \quad u_s(x) = U_{s,in} u_n^{(i)}(x) \quad (\text{A1.17})$$

For the zero net topcharge in the volume  $V_4$  (with, say, periodic boundary conditions and topcharges in the singular gauge) there are no global zero modes, and only quasizero modes, which e.g. for dilute gas of instantons of size  $\rho$ , have the Wigner semicircle distribution [15, 16]

$$\nu(\Lambda) = \frac{1}{\pi \bar{V}^2} (2N\bar{V}^2 - \Lambda^2)^{1/2} \Theta(2N\bar{V}^2 - \Lambda^2), \quad (\text{A1.18})$$

where the averaged value in the QCD instantonic vacuum [16]  $\bar{V}^2 \approx O\left(\frac{\rho^2}{V_4}\right)$ ,  $\bar{V}^2 = \frac{2\kappa^2}{N}$ ,  $\kappa \approx 0.14$  GeV.

In case with nonzero global topcharge  $Q$ , one obtains  $N_Q$  zero modes with  $\Lambda_s = 0$  in the sum (A1.16), and the sum (A1.16) is singular for  $m \rightarrow 0$ . This fact is exploited in Section 4 to exemplify the new mechanism for creation small fermion masses.

## Appendix 2

### Diagonalization of the mass matrix and the CKM parametrization

One can write the incident mass matrix obtained from the minima of wave functional (18) as

$$\hat{V} = \begin{pmatrix} \mu_1 & \mu_{12} & \mu_{13} \\ \mu_{12} & \mu_2 & \mu_{23} \\ \mu_{13} & \mu_{23} & \mu_3 \end{pmatrix} \quad (\text{A2.1})$$

and we shall test our assumption that nondiagonal elements  $\mu_{ij}$  due to kink solutions are close to "geometrical averages" of minima  $\mu_i$  and  $\mu_j$ , (Coherent Mixing Mechanism (CMM)), namely

$$\mu_{ij}^2 = \mu_i \mu_j + \Delta_{ij} = \mu_i \mu_j (1 + \eta_{ij}). \quad (\text{A2.2})$$

The eigenvalue equation for (A2.1) looks like

$$(\mu_1 - m)(\mu_2 - m)(\mu_3 - m) + 2\mu_{13}\mu_{12}\mu_{23} - \mu_{13}^2(\mu_2 - m) - \mu_{12}^2(\mu_3 - m) - \mu_{23}^2(\mu_1 - m) = 0. \quad (\text{A2.3})$$

We assume that the input masses  $\mu_i$  satisfy either condition I

$$\mu_1 < \mu_2 < \mu_3, \quad (\text{A2.4})$$

or more stringent condition II:

$$\mu_1 \ll \mu_2 \ll \mu_3. \quad (\text{A2.5})$$

Eq. (A2.3) with the use of (A2.2) can be expanded in powers of  $\eta_{ij} = \frac{\Delta_{ij}}{\mu_i \mu_j}$

$$m^3 - m^2\sigma + m\xi - \zeta = 0, \quad (\text{A2.6})$$

where  $\sigma = \sum_i \mu_i$ ,  $\xi = -\sum_{i \neq j} \Delta_{ij}$ ,  $\zeta = \mu_1 \mu_2 \mu_3 (-\frac{1}{4} \sum_{i \neq j} \eta_{ij}^2 + \frac{1}{2} \sum_{ij,lk} \eta_{ij} \eta_{lk})$ . Since we have connections between roots  $m_i$  and  $\sigma, \xi, \zeta$ , namely,

$$m_1 m_2 m_3 = \zeta; \quad \sum_{i \neq j} m_i m_j = \xi; \quad \sum_i m_i = \sigma \quad (\text{A2.7})$$

one can associate  $\zeta$  with the smallest root  $m_1$ ; then putting  $\zeta = 0$ , one obtains from (A2.6)  $m_1 = 0$  and for  $m_2, m_3$  one has the equation

$$m^2 - m\sigma + \xi = 0 \quad (\text{A2.8})$$

with the solutions:

$$m = \frac{\sigma}{2} \pm \sqrt{\frac{\sigma^2}{4} - \xi}, \quad m_2 = \frac{\xi}{\sigma} + \frac{\xi^2}{\sigma^3} + O\left(\left(\frac{\xi^3}{\sigma^5}\right)^3\right),$$

$$m_3 = \sigma - \frac{\xi}{\sigma} + O\left(\frac{\xi^2}{\sigma^3}\right). \quad (\text{A2.9})$$

To the first order in  $\zeta$  one obtains

$$m_1 = \frac{\zeta}{m_2 m_3}, \quad m_2 \cong \frac{\xi}{\sigma}, \quad m_3 \cong \sigma \quad (\text{A2.10})$$

and conditions  $m_i > 0$  yield  $\xi > 0, \zeta > 0$ . Using (A2.4) or (A2.5) one can estimate

$$m_1 \cong \frac{\zeta}{\xi}, \quad m_2 \approx \frac{\xi}{\mu_3}, \quad m_3 \approx \mu_3 + \mu_2 + \mu_1. \quad (\text{A2.11})$$

Thus we conclude that due to mixing the mass  $m_3$  does not move significantly from  $\mu_3$ , while the masses  $m_1, m_2$  can drastically decrease.

It is interesting to find out how the ratios of  $m_i$  may change, when  $\Delta_{ik}$  or  $\eta_{ik}$  are changing, i.e. we are interested in the motion of the eigenvalues  $m_i$  when mixing parameters  $\Delta_{ik}, \eta_{ik}$  are changing with  $\mu_i$  fixed. To this end we keep two of  $\eta_{ik}$  equal, e.g.  $\eta_{23} = \eta_{13} \equiv -\eta < 0$ , and  $\eta_{12} \equiv -\delta, |\delta| \ll \eta$ . In this case we have  $\zeta \cong \mu_1 \mu_2 \mu_3 \eta \delta$ ,  $\xi = \eta \mu_3 (\mu_1 + \mu_2)$ , and

$$m_1 \cong \delta \frac{\mu_1 \mu_2}{\mu_1 + \mu_2} \approx \delta \mu_1, \quad m_2 = \frac{\xi}{\sigma} \cong \frac{\eta \mu_3 (\mu_1 + \mu_2)}{\sigma}; \quad \frac{m_1}{m_2} \approx \frac{\delta}{\eta} \frac{\mu_1 \mu_2}{(\mu_1 + \mu_2)^2} \quad (\text{A2.12})$$

Thus one can see that  $\frac{m_1}{m_2}$  can be much smaller than  $\frac{\mu_1}{\mu_2}$  for  $\delta \ll \eta$ , while  $m_1$  can be made arbitrarily smaller than  $\mu_1$ , and  $m_2$  much smaller than  $\mu_2$  for small enough  $\delta$  and  $\eta (\delta \ll \eta)$ .

Let us now turn to the unitary matrix  $\widehat{W}$ , which diagonalizes the mass matrix (A2.1). At this point we can exploit the results of recent paper [18], where matrices  $W_u, W_d$  are given for any form of the matrix (A2.1).

To make our analysis more general, in (A2.1) we introduce also phases for matrix elements  $\mu_{ik}$ :  $\arg \mu_{ik} = \delta_{ik}$ ,  $ik = 12, 13, 23$  with the condition  $\delta_{12} - \delta_{13} + \delta_{23} = 0$ ; then the matrices which diagonalize the matrix  $\mu_{ik}$  with the eigenvalues  $m_1 < m_2 < m_3$  (so that  $\hat{\mu} = W^+ \hat{m} W$ ), can be readily written, using the general form of the unitary matrix from [18].

$$W = \begin{pmatrix} \frac{(\mu_2 - m_1)(\mu_3 - m_1) - |\mu_{23}|^2}{N_1}, & \frac{\mu_{13} \mu_{23}^* - \mu_{12}(\mu_3 - m_2)}{N_2}, & \frac{\mu_{12} \mu_{23} - \mu_{13}(\mu_2 - m_3)}{N_3} \\ \frac{\mu_{13}^* \mu_{23} - \mu_{12}^*(\mu_3 - m_1)}{N_1}, & \frac{(\mu_1 - m_2)(\mu_3 - m_2) - |\mu_{13}|^2}{N_2}, & \frac{\mu_{12}^* \mu_{13} - \mu_{23}(\mu_1 - m_3)}{N_3} \\ \frac{\mu_{12} \mu_{23}^* - \mu_{13}^*(\mu_2 - m_1)}{N_1}, & \frac{\mu_{12} \mu_{13}^* - \mu_{23}^*(\mu_1 - m_2)}{N_2}, & \frac{(\mu_1 - m_3)(\mu_2 - m_3) - |\mu_{12}|^2}{N_3} \end{pmatrix} \quad (\text{A2.13})$$

Here we have used the notations similar to those from [18].

$$\begin{aligned} N_1^2 &= (m_3 - m_1)(m_2 - m_1)[(\mu_2 - m_1)(\mu_3 - m_1) - |\mu_{23}|^2] \\ N_2^2 &= (m_3 - m_2)(m_2 - m_1)[(\mu_3 - m_2)(m_2 - \mu_1) + |\mu_{13}|^2] \\ N_3^2 &= (m_3 - m_2)(m_3 - m_1)[(m_3 - \mu_1)(m_3 - \mu_2) - |\mu_{12}|^2]. \end{aligned} \quad (\text{A2.14})$$

For the case of CMM the masses  $m_1 \approx \mu_1 \delta$ ,  $m_2 \approx (\mu_1 + \mu_2) \eta$ ,  $m_3 \approx \mu_1 + \mu_2 + \mu_3$ ,  $0 < \delta \ll \eta < 1$ , and one can estimate  $W_{31}$  in the limit  $\delta \rightarrow 0$ , ( $m_1 \rightarrow 0$ ),  $W_{31} \approx -\frac{1}{2} \sqrt{\frac{\mu_3}{m_3}} \sqrt{\frac{\mu_1}{\mu_1 + \mu_2}} \sqrt{\frac{\mu_2}{\mu_3}} \frac{\delta}{\eta}$ . One can use this limit,

$\delta \rightarrow 0$ , to simplify considerably the matrix  $W$ , as it is done in [18] if  $W_{13} = 0$ . In our case  $W$  has the same form as obtained in [18] for  $W^+$ , so that in our notations we can write as<sup>7</sup>

$$W = \begin{pmatrix} c_\alpha, & -s_\alpha c_\beta e^{i\delta_{12}}, & s_\alpha s_\beta e^{i\delta_{13}} \\ -s_\alpha e^{-i\delta_{12}}, & -c_\alpha c_\beta & , c_\alpha s_\beta e^{i\delta_{23}} \\ 0, & s_\beta e^{-i\delta_{23}}, & c_\beta \end{pmatrix} \quad (\text{A2.15})$$

where we have denoted

$$c_\alpha = \sqrt{\frac{m_2 - \mu_1}{m_2 - m_1}}, \quad s_\alpha = \sqrt{1 - c_\alpha^2}, \quad c_\beta = \sqrt{\frac{m_3 - \mu_2}{2m_3 - \mu_2 - \mu_3}}, \quad s_\beta = \sqrt{1 - c_\beta^2}. \quad (\text{A2.16})$$

Note, that the matrix (A2.15) contains 4 independent parameters:  $s_\alpha, s_\beta$  and two phases  $\delta_{ik}$ , as it should be for the  $3 \times 3$  unitary matrix. This form of  $W, W^+$  is assumed both for  $u$  and  $d$  sectors and the entries will be denoted as  $c_\alpha^u, s_\alpha^u, c_\beta^u, s_\beta^u$  and the same for  $d$ . As a result one can construct the CKM matrix  $V_{CKM} \equiv W_u W_d^+$ .

Writing  $V_{CKM} = \{V_{ik}\}$ , one has for matrix elements.

$$\begin{aligned} V_{11} &= c_\alpha^u c_\alpha^d + c_\beta^u c_\beta^d s_\alpha^u s_\alpha^d e^{i\Delta_{12}} + s_\alpha^u s_\beta^u s_\alpha^d s_\beta^d e^{i\Delta_{13}}, \\ V_{12} &= s_\alpha^u c_\alpha^u c_\alpha^d c_\beta^d e^{i\delta_{12}^u} - c_\alpha^u s_\alpha^d e^{i\delta_{12}^d} + s_\alpha^u s_\beta^u c_\alpha^d c_\beta^d e^{i\delta_{13}^u - i\delta_{23}^d} \\ V_{13} &= s_\alpha^u s_\beta^u c_\beta^d e^{i\delta_{13}^u} - s_\beta^d s_\alpha^u c_\beta^u e^{i(\delta_{12}^u + \delta_{23}^d)} \\ V_{21} &= -s_\alpha^u c_\alpha^d e^{-i\delta_{12}^u} + c_\alpha^u c_\beta^u c_\beta^d s_\alpha^d e^{-i\delta_{12}^d} + c_\alpha^u s_\beta^u s_\alpha^d s_\beta^d e^{i\delta_{23}^u - i\delta_{13}^d} \quad (\text{A2.17}) \\ V_{22} &= s_\alpha^u s_\alpha^d e^{-i\Delta_{12}} + c_\alpha^u c_\beta^u c_\alpha^d c_\beta^d + c_\alpha^u c_\alpha^d s_\beta^u s_\beta^d e^{i\Delta_{23}}, \\ V_{23} &= c_\alpha^u c_\beta^d s_\beta^u e^{i\delta_{23}^u} - c_\alpha^u c_\beta^u s_\beta^d e^{i\delta_{23}^d}, \\ V_{31} &= c_\beta^u s_\alpha^d s_\beta^d e^{-i\delta_{13}^d} - c_\beta^d s_\beta^u s_\alpha^d e^{-i\delta_{23}^u - i\delta_{12}^d}, \end{aligned}$$

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<sup>7</sup>Note that the form (A2.15) does not contain the terms  $O(\eta), O(\delta)$ , which are present in (A2.13).

$$V_{32} = -s_\beta^u c_\alpha^d c_\beta^d e^{-i\delta_{23}^u} + c_\alpha^d c_\beta^u s_\beta^d e^{-i\delta_{23}^d},$$

$$V_{33} = s_\beta^u s_\beta^d e^{-i\Delta_{23}} + c_\beta^u c_\beta^d.$$

Here notation is used  $\Delta_{ik} \equiv \delta_{ik}^u - \delta_{ik}^d$ .

One can also easily calculate the Jarlskog invariant  $J$ ,

$$J = \text{Im}(V_{23} V_{33}^* V_{21}^* V_{31}) \quad (\text{A2.18})$$

Using (A2.17) one can write the leading term ( $c_\beta \approx c_\alpha = 1, s_\beta \ll 1, s_\alpha \ll 1$ ) in the form

$$J = s_\alpha^u s_\alpha^d |s_\beta^d - s_\beta^u e^{i\Delta_{23}}|^2 \sin \Delta_{12} \quad (\text{A2.19})$$

## Appendix 3

### Coherent mixing in four generations

Here at first we simplify the problem, neglecting the phases  $\delta_{ij}$  in the  $4 \times 4$  mass matrix, taken in the form (45), and define the eigenvalues  $m_i, i = 1, 2, 3, 4$  from the quartic equation

$$m^4 - a_1 m^3 + a_2 m^2 - a_3 m + a_4 = 0, \quad (\text{A3.1})$$

where the coefficients  $a_i$  can be expressed either via the eigenvalues:

$$a_1 = \sum_{i=1}^4 m_i, \quad a_2 = \sum_{i \neq j=1}^4 m_i m_j, \quad a_3 = \sum_{i \neq j \neq k}^4 m_i m_j m_k, \quad a_4 = \prod_{i=1}^4 m_i, \quad (\text{A3.2})$$

or through the mass matrix coefficients:  $\mu_{ki} = \mu_{ik} = \sqrt{\mu_i \mu_k (1 + \eta_{ik})}$ . Expanding to the lowest power of  $\eta_{ik}$ , one obtains

$$a_1 = \sum_{i=1}^4 \mu_i, \quad a_2 = - \sum_{i \neq k} \mu_i \mu_k \eta_{ik}. \quad (\text{A3.3})$$

$$a_3 = \prod_{i=1}^4 \mu_i \sum_{n=1}^4 \frac{1}{\mu_n} \sum_{i,k,l \neq n} \left( \frac{1}{2} \eta_{il} \eta_{kl} - \frac{1}{4} \eta_{ik}^2 \right). \quad (\text{A3.4})$$

$$a_4 = \frac{1}{4} \prod_{i=1}^4 \mu_i \left\{ \sum_{i \neq k \neq l} \eta_{ik} \eta_{kl} \eta_{li} - \sum_{i \neq s \neq k \neq l} \eta_{ik} \eta_{kl} \eta_{ls} + \sum_{i \neq j \neq l \neq k} \eta_{ij}^2 \eta_{lk} \right\} = \quad (\text{A3.5})$$

$$\begin{aligned} &= \frac{1}{4} \prod_{i=1}^4 \mu_i \left\{ \eta_{12} \eta_{13} \eta_{23} + \eta_{14} (\eta_{23}^2 - \eta_{13} \eta_{23} - \eta_{12} \eta_{23}) + \right. \\ &\quad \eta_{24} (\eta_{13}^2 - \eta_{12} \eta_{13} - \eta_{13} \eta_{23}) + \eta_{34} (\eta_{12}^2 - \eta_{12} \eta_{13} - \eta_{12} \eta_{23}) + \\ &\quad \eta_{23} (\eta_{14}^2 - \eta_{14} \eta_{24} - \eta_{14} \eta_{34} + \eta_{34} \eta_{24}) + \eta_{13} (\eta_{24}^2 - \eta_{24} \eta_{34} - \eta_{14} \eta_{24} + \eta_{34} \eta_{14}) + \\ &\quad \left. + \eta_{12} (\eta_{34}^2 - \eta_{14} \eta_{34} - \eta_{24} \eta_{34} + \eta_{24} \eta_{14}) \right\}. \end{aligned} \quad (\text{A3.6})$$

In the last form (A3.6) the terms are ordered according to the power of  $\eta_{i4}$ , which can tend to zero.

To simplify coefficients and establish a connection with the case of 3 generations, we assume that only three new elements are nonzero in the  $4 \times 4 \mu_{ik}$  as compared to  $3 \times 3 \mu_{ik}$ , namely,  $\mu_4, \eta_{34} = \eta_{24} = -\bar{\eta}, \bar{\eta} > 0$  in addition to considered before  $\eta_{13} = \eta_{23} = -\eta, \eta_{12} = -\delta$ .

Then to the lowest order one has

$$a_2 = \mu_1 \mu_2 \delta + \mu_3 (\mu_1 + \mu_2) \eta + \mu_4 (\mu_2 + \mu_3) \bar{\eta}. \quad (\text{A3.7})$$

$$a_3 = \mu_1 \mu_2 \mu_3 \eta \delta - \frac{1}{4} \mu_1 \mu_2 \mu_4 (\delta - \bar{\eta})^2 - \frac{1}{4} \mu_1 \mu_3 \mu_4 (\eta - \bar{\eta})^2 + \mu_2 \mu_3 \mu_4 \eta (\bar{\eta} - \frac{\eta}{4}), \quad (\text{A3.8})$$

$$a_4 = \frac{3}{4} \prod_{i=1}^4 \mu_i \delta \eta \bar{\eta}. \quad (\text{A3.9})$$

In the case, when  $\bar{\eta} > \frac{\eta}{4}$  and  $\mu_4 > \mu_3 > \mu_2 > \mu_1$ , all coefficients  $a_i, i = 1, 2, 3, 4$  are positive, yielding positive eigenvalues  $m_i$ . In the limit  $\delta \rightarrow 0, \eta \rightarrow 0$  both  $a_3, a_4$  vanish and for two largest eigenvalues one has

$$m_4, m_3 = \frac{a_1}{2} \pm \sqrt{\frac{a_1^2}{4} - a_2}, \quad (\text{A3.10})$$

$$m_4 = \sigma - \frac{\mu_3 \mu_4 \bar{\eta}}{\sigma}, \quad \sigma = \sum_{i=1}^4 \mu_i, \quad (\text{A3.11})$$

$$m_3 = \frac{\mu_3 \mu_4 \bar{\eta}}{\sigma}. \quad (\text{A3.12})$$

Thus again, as in the case of three generations, the largest mass  $m_4$  is slightly larger than  $\mu_4$ , whereas  $m_3 \approx \bar{\eta}\mu_3$  can be strongly shifted from the input value  $\mu_3$ . For  $m_1, m_2$  one obtain an estimate

$$m_2 \cong \frac{a_3}{m_3 m_4} \cong \frac{a_3}{\bar{\eta}\mu_3\mu_4} = \frac{\mu_2\eta(\bar{\eta} - \eta/4)}{\bar{\eta}}, \quad m_1 = \frac{a_4}{m_2 m_3 m_4} \cong \frac{\frac{3}{4}\mu_1\delta\bar{\eta}}{\bar{\eta} - \eta/4}. \quad (\text{A3.13})$$

Now we have to find the unitary matrix  $W$ , which diagonalizes the  $4 \times 4$  matrix  $\mu_{ik}$ ; for that we impose two conditions

$$\hat{W}^+ \hat{W} = \hat{1}, \quad W_{ki}^* W_{kl} = \delta_{il} \quad (\text{A3.14})$$

$$\hat{W}^+ \hat{m} \hat{W} = \hat{\mu}, \quad (\text{A3.15})$$

where  $\hat{m} = \text{diag.}(m_1, m_2, m_3, m_4)$ .

In CMM the simplest  $(4 \times 4)$  form which is a natural extension of the  $(3 \times 3)$  matrix (A3.15) is

$$W = \begin{pmatrix} c_\alpha, & -s_\alpha c_\beta e^{i\delta_{12}}, & c_4 s_\alpha s_\beta e^{i\delta_{13}}, & s_4 s_\beta s_\alpha e^{i\delta_{14}} \\ -s_\alpha e^{-i\delta_{12}}, & -c_\alpha c_\beta, & c_4 s_\beta c_\alpha e^{i\delta_{23}}, & s_4 s_\beta c_\alpha e^{i\delta_{24}} \\ 0, & s_\beta e^{-i\delta_{23}}, & c_4 c_\beta, & s_4 c_\beta e^{i\delta_{34}} \\ 0, & 0, & -s_4 e^{-i\delta_{34}}, & c_4 \end{pmatrix}. \quad (\text{A3.16})$$

The phases  $\delta_{ik}$  satisfy conditions

$$\delta_{24} = \delta_{23} + \delta_{34}, \quad \delta_{14} = \delta_{13} + \delta_{34}, \quad \delta_{14} - \delta_{24} = \delta_{12}. \quad (\text{A3.17})$$

One can notice that only two of these conditions are independent of the old,  $3 \times 3$  condition  $\delta_{12} - \delta_{13} + \delta_{23} = 0$ . Here  $s_4^2 + c_4^2 = 1$ , so that in (A3.16) one has two new parameters, e.g.  $s_4$  and  $\delta_{34}$ .

One can check, that the unitarity condition of  $W$ , Eq. (A3.14), is satisfied, while (A3.15) yields approximately

$$\begin{aligned} \mu_4 &\cong c_4^2 m_4 + s_4^2 c_\beta^2 m_3 + \dots \\ \mu_3 &\cong m_4 s_4^2 + m_3 c_4^2 c_\beta^2 + \dots \\ \mu_2 &\cong m_3 s_\beta^2 + m_2, \quad \mu_1 \cong m_2 s_\alpha^2 + m_1 c_\alpha^2. \end{aligned} \quad (\text{A3.18})$$

From (A3.18) one can see that  $\mu_4 < m_4$  and  $\mu_i > m_i, i = 1, 2, 3$  and the relations (A3.11-A3.13) are approximately satisfied.

From (A3.15) one can find  $c_\alpha, c_\beta, c_4 (s_i = \sqrt{1 - c_i^2}, i = \alpha, \beta, 4)$ :

$$c_\alpha^2 = \frac{m_2 - \mu_1}{m_2 - m_1}, \quad c_\beta^2 = \frac{m_3 - \mu_2}{m_3 + \mu_1 - m_1 - m_2}, \quad c_4^2 \cong \frac{\mu_4 - m_3 + \mu_2}{m_4 - m_3 + \mu_2}. \quad (\text{A3.19})$$